Small-world networks: Links with long-tailed distributions

S. Jespersen[1,2] and A. Blumen[2]
[1]*Institute of Physics and Astronomy, University of Aarhus, DK-8000 Århus C, Denmark*
[2]*Theoretische Polymerphysik, Universität Freiburg, D-79104 Freiburg im Breisgau, Germany*
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Small-world networks (SWN), obtained by randomly adding to a regular structure additional links (AL), are of current interest. In this paper we explore (based on physical models) a new variant of SWN, in which the probability of realizing an AL depends on the chemical distance between the connected sites. We assume a power-law probability distribution and study random walkers on the network, focusing especially on their probability of being at the origin. We connect the results to Lévy flights, which follow from a mean-field variant of our model.

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I. INTRODUCTION

Recently, a lot of interest has centered on the so-called small-world networks (SWN) [1,2] where an underlying regular lattice is supplemented with additional links (bonds), a fact which drastically reduces the minimal distances (the so-called chemical distances in the fractal literature) between pairs of points on the lattice [1,3–8]. This question is of the utmost importance for general network structures, e.g., Internet links [9], and for the spreading of diseases [3,10–13], topics which depend on the minimal distances. On the other hand, other questions are possible for such structures; for example, random transport [14,15]; this requires solving diffusion-type problems, which are mathematically described by the Laplacian on the structure [16] and the corresponding eigenvalues and eigenvectors [17]. Examples of such problems are anomalous transport of charges and of excitations over networks [18,19]. Most recent SWN studies center on a one-dimensional chain supplemented with additional links (AL), which connect sites that are arbitrarily far from each other on the underlying lattice. While this is the simplest SWN that can be envisaged, there are situations in which links between distant sites occur naturally; however, their lengths are then not necessarily uniformly distributed: Considering a polymer chain in solution, monomers which are far apart along the backbone can be quite close to each other in real space so that, for instance, energy transfer over the structure may take cross cuts along sites close to each other in space [18]. Now the probability of having such close monomer pairs is related to the return to the origin of random walks, possibly under self-avoiding constraints. In this case, the probability $p(l)$ that two sites far apart along the backbone come close together in space is approximately an inverse power law of their mutual chemical distance l [20]:

$$p(l) = \frac{a}{l^{\alpha}}. \tag{1}$$

In this work, we will focus on networks constructed as in the SWN case, while, however, letting the additional bonds be distributed according to Eq. (1). We call these structures generalized small-world networks (GSWN). Clearly, the original SWN is recovered from the GSWN by letting $\alpha \to 0$. On the other hand, in GSWNs with $\alpha \gg 1$, practically only sites that are already very close on the chain get to be connected; such GSWNs have (apart from disorder) properties similar to the underlying regular lattice. Most interesting are GSWNs with $0 \leqslant \alpha \leqslant 3$, which show a wealth of features, because of the long-range character of the additional links.

In this paper, we will study random walks over GSWN, and especially the probability $P_0(t)$ of the walker being (still or again) at the origin; as discussed in previous works [14,21], this quantity reflects many of the properties of the density of eigenvalues of the underlying structure, and is easily obtainable by very effective, easy-to-program numerical procedures.

Our paper is structured as follows. In the next section (Sec. II) we discuss the construction of GSWN in more detail. In Sec. III, we study the behavior of random walkers on GSWN. We find that for α well below 2, we have a behavior qualitatively similar to that of walkers over the SWN. However, for α larger than 2 we move towards another regime, quite reminiscent of random walks on regular lattices. The transition appears to happen around $\alpha \approx 2$, which prompts us to consider transient versus recurrent walks in Sec. IV. In this section we determine analytically $P_0(t)$ for a mean-field variant of the GSWN model, which we then compare with the numerical findings $P_0(t)$ on GSWN. It turns out that the mean-field approach is related to Lévy flights and Lévy walks. Finally, we close our paper by summarizing our conclusions in Sec. V.

II. CONSTRUCTION OF GSWN

The construction of GSWN at first follows the SWN procedure closely: We start from a ring of N sites (i.e., a closed, regular one-dimensional lattice). Then we consider each site consecutively, and let it sprout with probability q an additional bond, which connects it to another site; see Fig. 1. We now let $p(l)$ in Eq. (1) be the conditional probability that this bond gets attached to a site at the (minimal) chemical distance l from the sprouting site, measured along the ring. Here, given our periodic boundary conditions, the chemical distance l lies between 1 and int$(N/2)$, where int(x) denotes the largest integer X such that $X \leqslant x$ (see Fig. 1). Note that through $p(l)$ our model differs from the standard SWN,

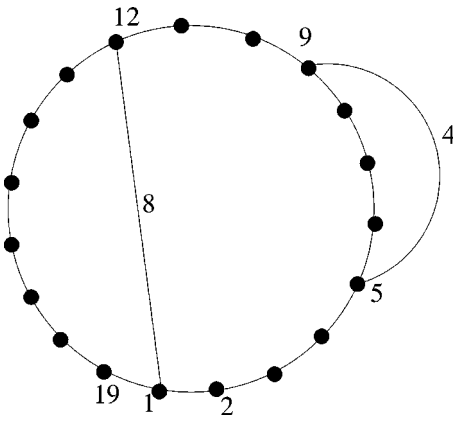

FIG. 1. Illustration of the small-world network. In this example, two additional links are added, with the corresponding distances given in the figure.

where no l dependence is accounted for. In Eq. (1), the constant a normalizes the distribution so that

$$2\sum_{l=1}^{\mathrm{int}(N/2)} al^{-\alpha}=1. \tag{2}$$

Now the exponent α in Eq. (1) is a parameter of the model and will be varied in the following. For a finite system we can choose it freely. In fact, the choice $\alpha=0$, i.e., $p(l)=1/(N-1)$, recovers one of the basic constructions of the SWN, by which connections to all sites but the source are equiprobable. For an infinite network, on the other hand, care has to be taken; in order to keep Eq. (1) normalizable, one has to have $\alpha>1$.

We now turn to the basic procedure, in which the structure of the model enters through its connectivity matrix $\mathbb{A}$. Now $\mathbb{A}$ is defined as follows: The off-diagonal elements of $\mathbb{A}$, namely A_{ij} with $i\neq j$, equal *minus* the number of links between the sites i and j of the network. The diagonal elements A_{ii} count the total number of bonds connected to i. Hence the connectivity matrix is symmetric and one has $\Sigma_i A_{ij}=0$. Furthermore, $\det(\mathbb{A})=0$ and exactly one eigenvalue of $\mathbb{A}$, say E_1, vanishes. One should note that from the spectrum of the $\mathbb{A}$ matrix one can determine, e.g., the diffusion and vibrational properties of the structure, as well as its behavior in external fields [17,21]. The spectrum of $\mathbb{A}$ for the SWN ($\alpha=0$) has been recently studied by Monasson [16]; among his findings was the existence of a pseudogap in the density of states, a property which affects the long-time diffusion properties [21]. We note that $\mathbb{A}$ can be viewed as arising from two sources: One term, $\mathbb{A}^{(1)}$, is deterministic and is due to the underlying regular lattice (here the ring). Another one, $\mathbb{A}^{(2)}$, is stochastic and arises from the randomly added links. Thus $\mathbb{A}=\mathbb{A}^{(1)}+\mathbb{A}^{(2)}$. Now formally $A^{(1)}_{ii\pm1}=-1$, $A^{(1)}_{ii}=2$, and $A^{(1)}_{ij}=0$ otherwise, where we identify site $N+1$ with 1 (periodic boundary conditions). The entries in $A^{(2)}_{ij}$ are random, and for $i\neq j$ they are equal to 0, -1, or -2. In fact, letting l be the chemical distance between i and j ($i\neq j$), one has for the probability $P_l(c)$ that $A^{(2)}_{ij}=-c$,

$$P_l(c)=\binom{2}{c}(qal^{-\alpha})^c(1-qal^{-\alpha})^{2-c}. \tag{3}$$

The diagonal elements are, as before, determined from the requirement that $\Sigma_i A^{(2)}_{ij}=0$. We close on a small note of caution by remarking that, due to our prescription, even for $j\in\{1,\ldots,i-1\}$, the elements $A^{(2)}_{ij}$ are not independent of each other. Thus if $A^{(2)}_{ij}=-2$ for $i\neq j$, then for $k\notin\{i,j\}$ one cannot have $A^{(2)}_{ik}=-2$; a nondiagonal element having a value of -2 implies both for j and for k that one of their additional bonds has started at i. By construction, however, i can only be the source of one additional bond. For decreasing q and increasing N, however, we expect such correlations between the $A^{(2)}_{ij}$ to be less and less important.

III. PROBABILITY OF BEING AT THE ORIGIN

As a simple dynamical problem on the underlying structure, we focus now on the probability for a random walker to be (still or again) at the origin of its walk at a later time. This quantity is fundamental for fractal lattices, where it leads directly to the spectral (harmonic) dimension [15], a quantity of much importance [22]. As we have shown in a previous work, determining this quantity through a numerical cellular automaton procedure is quite straightforward and very revealing for SWN [14]. We look at the probability $P(i,t|m)$ for the walker to be at site i at time t, given that it started at site m at time $t=0$. One notes first that $P(i,t|m)$ obeys the following master equation:

$$\frac{\partial P(i,t|m)}{\partial t}=-\sigma\sum_{j=1}^{N} A_{ij}P(j,t|m), \tag{4}$$

where σ is a transition rate. In vector notation $\mathbf{P}^{(m)}(t)\equiv(P(1,t|m),\ldots,P(N,t|m))$ this relation reads

$$\frac{\partial\mathbf{P}^{(m)}(t)}{\partial t}=-\sigma\mathbb{A}\mathbf{P}^{(m)}(t) \tag{5}$$

and has the formal solution

$$\mathbf{P}^{(m)}(t)=\exp(-\sigma\mathbb{A}t)\mathbf{P}^{(m)}(0). \tag{6}$$

Now the initial condition is $\mathbf{P}^{(m)}(0)=(0,\ldots,1,\ldots,0)$ with a single nonzero element at m. The probability that the walker is again at m at time t reads

$$P(m,t|m)=[\mathbf{P}^{(m)}(t)]_m=\sum_j [\exp(-\sigma\mathbb{A}t)]_{mj}[\mathbf{P}^{(m)}(0)]_j$$
$$=[\exp(-\sigma\mathbb{A}t)]_{mm}. \tag{7}$$

This expression simplifies by averaging over all starting points, since then

$$\frac{1}{N}\sum_{m=1}^{N} P(m,t|m)=\frac{1}{N}\mathrm{Tr}[\exp(-\sigma\mathbb{A}t)]=\frac{1}{N}\sum_{i=1}^{N} e^{-E_i\sigma t} \tag{8}$$

holds, where Tr denotes the trace operation and E_i with $1\leqslant i\leqslant N$ are the eigenvalues of the (symmetric) connectivity matrix $\mathbb{A}$. Note that in Eq. (8), because of the averaging over all initial points, only the eigenvalues enter. Furthermore, one can now readily average over different realizations, obtaining (since $E_1\equiv0$)

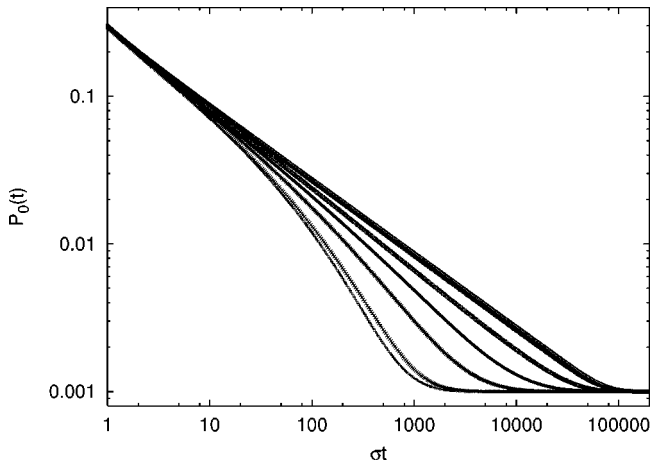

FIG. 2. The probability of being at the origin, $P_0(t)$, for $q=0.05$ and $\alpha=3,2,1.5,1.25,1,0.5$, and 0.0 from above. The curves for $\alpha=2$ and $\alpha=3$ are hardly distinguishable in the figure.

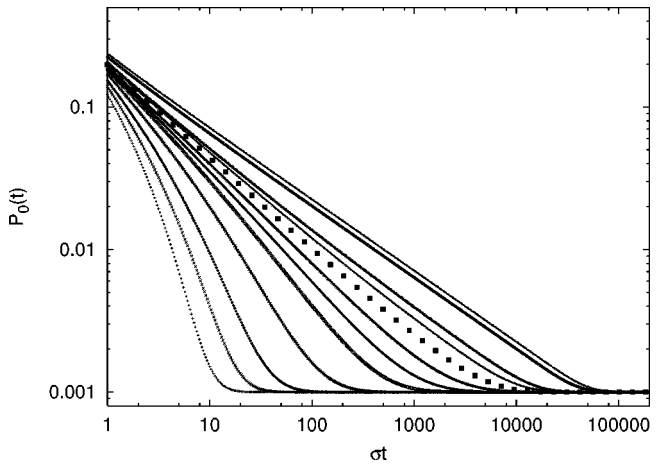

FIG. 3. Same as Fig. 2, but for a choice of $q=0.8$. From above one has $\alpha=5.0,3.0,2.2,2.1,2.0,1.9,1.8,1.7,1.5,1.0,0.5$, and 0.0; the curve for $\alpha=2$ is indicated by dots.

$$P_0(t)\equiv\frac{1}{N}+\frac{1}{N}\left\langle\sum_{i=2}^{N}e^{-E_i\sigma t}\right\rangle=\frac{1}{N}+\int\rho(E)e^{-Et}dE \quad (9)$$

with $\rho(E)$ being the spectral density for $E>0$.

We turn now to our calculations, by which we determine numerically $P_0(t)$ for different choices of α and q. For systems of size $N=1001$, we construct the connectivity matrix, diagonalize it, and employ Eq. (8) to evaluate $P_0(t)$. We use for each choice of α and q 100 realizations to average over the structural disorder. In Fig. 2, we display on double logarithmic scales $P_0(t)$ versus the dimensionless time σt for $q=0.05$ and for α ranging from 0 to 3. First we note that for very long times, $P_0(t)$ reaches the constant value $1/N$, which arises due to the eigenvalue $E_1=0$. Increasing the size of the small-world network (i.e., N) pushes the long-time plateau to lower values, but, as we have shown in an earlier work for $\alpha=0$ [14], leaves the $P_0(t)$ curves above the plateau practically unaffected. This is also what we find here for general α; this allows us to infer the qualitative features of $P_0(t)$ in the limit $N\to\infty$.

Turning now to the discussion of the results, we note first that for $\alpha=0$ they agree perfectly with our previous small-world network analysis [14], which was based not on the diagonalization of $\mathbb{A}$ but on a cellular automaton method. The decay of $P_0(t)$ follows at early times a power law, which turns at later times into a stretched exponential behavior. Asymptotically, the decay obeys to leading order the form $\exp(-Ct^{1/3})$ [14,21] (with C a constant), which follows from the spectral density of Ref. [16], $\rho(E)\sim E^{-1/2}\exp(-C'E^{-1/2})$. Note that in Fig. 2, the curves flatten with increasing α, a sign that with growing α, a walker is less prone to go far away from its starting site. At early times $P_0(t)$ is little affected by variations in α, since at very short times it does not matter whether the AL bring the walkers very far away or not. The transition to pure power-law behavior appears to happen roughly around $\alpha=2$.

Moving on to larger q to examine whether this transition depends on q, we plot in Fig. 3 the results for $q=0.8$. In this case there are more AL, and the results are more sensitive to the value of α: in Fig. 3 the cases $\alpha=3.0$ and $\alpha=5.0$ are easily distinguished. However, increasing α further does not change the curves significantly. The curves are also more spread out in the short-time domain than in the case $q=0.05$. This is due to the fact that the quasi-one-dimensional behavior of the walk is mainly felt on distances of the order of $1/q$, this being a measure of the mean distance between branching points [14]. We hasten to note that for very small times (not displayed in Fig. 3), the curves for different α do coincide. Despite these differences, the qualitative behavior of the curves in Fig. 2 and Fig. 3 is comparable. Furthermore, the crossover behavior of $\alpha=2$ (shown as a dotted line) appears even more clearly in Fig. 3: the curves with $\alpha>2$ follow power-law decays closely, while the curves for $\alpha<2$ are partly concave, thus displaying a faster-than-power-law decay.

IV. TYPOLOGY OF RANDOM WALKS

Let us briefly recall some terminology from the theory of random walks. A random walk is said to be recurrent if it returns with probability 1 to the origin at some later time. Otherwise the walk is called transient. For a walk to be transient requires an infinite system, because in finite systems all walks (disregarding situations with traps, mortal walkers, etc.) are recurrent. On homogeneous lattices, a walk is transient if and only if

$$I\equiv\int_0^\infty dt\,P_0(t) \quad (10)$$

is finite. In line with this, we could expect walks on the GSWN with small α to be transient, given that for a stretched exponential behavior ($\beta>0$)

$$I\sim\int^\infty dt\exp(-Ct^\beta)<\infty. \quad (11)$$

Moreover, in the opposite limit of large α, we observe that $P_0(t)$ follows a power-law decay with the exponent being nearly $-1/2$. It follows that for large α we have $I=\infty$, an indication that the walk is recurrent. As we discuss in detail in the following, for walks on an infinite regular linear chain whose steps are long-ranged and obey Eq. (1) for the step

lengths, the transition between recurrence and transience occurs at $\alpha=2$ [24]. It is now tempting to try to explain our findings of Sec. III along such lines. Such a connection is achieved by replacing the random small-world network structure under investigation here by a regular one (a mean-field-type approach), and letting the transition rates reflect the underlying statistics of the links [23]. In this way the probability of taking a step of length $l>1$ is proportional to $l^{-\alpha}$. However, as we show in the following, this regularizing approach is not particularly successful, since it does not describe $P_0(t)$ well for small and moderate α.

We start now from the so-called Riemann walks [24], which are symmetric random walks on the linear chain, where each step of the walk can extend over the length l with probability

$$\pi(l)\sim l^{-\alpha}, \quad \alpha>1. \tag{12}$$

Such walks are recurrent for $\alpha\geqslant 2$ and transient for $\alpha<2$ [24]. Riemann walks are examples from the more general class of Lévy flights and Lévy walks [25–27]. Turning now to the problem of averaging both the GSWN structures and the random walks over them, we simply replace in Eq. (4) $\mathbb{A}^{(2)}$ (remember that $\mathbb{A}=\mathbb{A}^{(1)}+\mathbb{A}^{(2)}$) by its average $\langle\mathbb{A}^{(2)}\rangle$ over all GSWN. For the averaging we may use Eq. (3) and obtain

$$\langle A_{ij}^{(2)}\rangle = 2al^{-\alpha}q \equiv c(l), \tag{13}$$

where l is the chemical distance between i and j. By doing this we have now as connectivity matrix $\mathbb{A}=\mathbb{A}^{(1)}+\langle\mathbb{A}^{(2)}\rangle$, whose eigenvalues E_k are readily found; they read (for N odd)

$$E_k = 2-2\cos(2\pi k/N)+2\sum_{j=1}^{(N-1)/2}[1-\cos(2\pi kj/N)]c(j), \tag{14}$$

where $k=0\cdots N-1$. Now Eq. (14) can be used to determine numerically $P_0(t)$ via Eq. (9).

In Fig. 4, we compare for $N=1001$, $q=0.05$, and $\alpha=3$, $\alpha=1.5$, and $\alpha=0$ the results of the two approaches. For $\alpha=3$ the two methods lead to a nice agreement; it seems that for α around or larger than 3 the fluctuations due to the disorder play only a minor role. On the other hand, as exemplified by $\alpha=1.5$ and $\alpha=0$, for α below 2 the mean-field approach leads to $P_0(t)$ forms that are quite different from those obtained in Sec. III.

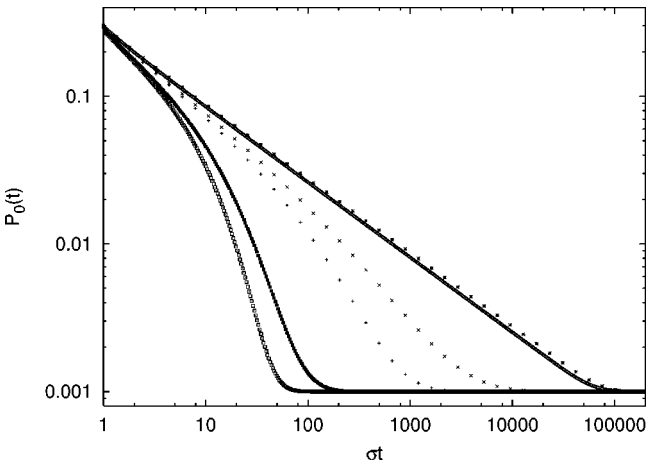

FIG. 4. Comparison of the mean-field theory prediction (full curves) with the numerical data (dotted curves) for $q=0.05$. The values of α are, from above, $\alpha=3$, $\alpha=1.5$, and $\alpha=0$. Good agreement is found only for large α, here $\alpha=3$.

V. CONCLUSION

In this work, we have studied a new variant of the small-world network (SWN) model which takes into account the fact that the probability of adding links can depend on the chemical distance between the connected sites. Exemplary, here we have taken the probability distribution to be a power law (with exponent α) of the chemical distance, see Eq. (1). We have focused on random walks and especially on the probability $P_0(t)$ of a random walker to be at its origin. Depending on the value of α, we have found qualitatively different behaviors. Specifically, we found clues indicating that in the infinite system limit, random walks on GSWN may change from being transient to being recurrent, as α crosses the marginal value of 2 from below. Moreover, we have shown that our model is related to Lévy flights and to Riemann walks. We also found that a simple mean-field regularization of the GSWN problem gives poor results for small α. Overall, it follows that GSWN with $\alpha<2$ are objects whose dynamical properties differ significantly from those of regular lattices.

ACKNOWLEDGMENTS

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